Ma 116 – Exam I February 6, 2001

Name: ***Solutions*** ID:

Check your lecture: ☐ Miller ☐ Gilman ☐ Tabanjeh (11:00) ☐ Tabanjeh (12:00)

Closed book and closed notes.
Calculators and laptops are not allowed.

Show all of your work. Answers without supporting work may not receive credit.

Pledge and Sign:

Problem I.

1. (7pts) State the *integral test* for an infinite series of the form $\sum_{n=1}^{\infty} a_n$.

   Solution: Suppose $a_n = f(n)$ where the function f is continuous, positive, and decreasing on $[1, \infty)$.

   (a) If $\int_1^{\infty} f(x)\, dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\int_1^{\infty} f(x)\, dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

2. (8pts) Apply the integral test to determine if the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

   Solution:

$$\begin{aligned}\int_2^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\, dx \\ &= \lim_{t\to\infty} \int_{\ln 2}^{t} \frac{du}{u} \\ &= \lim_{t\to\infty} \ln u \Big|_{\ln 2}^{t} = \infty\end{aligned}$$

   It follows from the theorem in part 1 that this series diverges.

Problem II. Determine which of the following series converge. Support your answer.

1. (5pts) $-2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \cdots$

   Solution: The series is absolutely convergent since the series of absolute values is a geometric series with $r = 1/3$, or prove directly using Ratio Test, $|a_{n+1}/a_n| = 1/3$ for all n.

   Also converges by alternating series test since $2/3^n$ is decreasing, and goes to 0 as $n \to \infty$.

2. (5pts) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$

   Solution: Diverges since this is a p-series with $p = 1/2 \leq 1$, or prove directly using the integral test,

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} 2\left(\sqrt{t} - 1\right) = \infty.$$

3. (5pts) $1 + \frac{3}{2} + \frac{9}{4} + \frac{81}{16} + \frac{243}{64} + \cdots + \frac{3^n}{2^n} + \cdots$

   Solution: From the general term $(3/2)^n$, the series is geometric with $r = 3/2 \geq 1$ and therefore the series diverges (or by the Ratio Test $|a_{n+1}/a_n| = 3/2 > 1$).

   Note: There was an error in writing the first few terms: $n = 3$ is missing and $n = 4$ has 2^5 in the denominator.

4. (5pts) $\sum_{n=2}^{\infty} \frac{1}{(n^2 - 2)}$

   Solution:
   Use Limit Comparison Test with the convergent series $\sum 1/n^2$,

$$\lim_{n\to\infty} \left(\frac{1}{n^2 - 2}\right)\left(\frac{n^2}{1}\right) = 1 \implies \sum \frac{1}{n^2 - 2} \text{ converges}$$

   Comparison Test with the convergent series $\sum 1/2n^2$,

$$\frac{1}{n^2 - 2} < \frac{1}{2n^2} \text{ for } n \geq 2 \implies \sum \frac{1}{n^2 - 2} \text{ converges}$$

Problem III. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

1. (7pts) Find the radius of convergence and the interval of convergence.

Solution: Apply the Ratio Test to determine the radius of convergence,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{|x|^{n+1}}{n+1} \frac{n}{|x|^n}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| < 1 \implies R = 1.$$

At $x = -1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test.

At $x = +1$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test (p-series with $p = 1$).

$\implies$ Interval of convergence is $I = [-1, 1)$.

2. (8pts) Find the sum of this series. (*Hint*: Consider the derivative of the power series.)

Solution: Let $f(x)$ denote the sum of the series on its interval of convergence.

Differentiate $f(x) = \sum x^n/n$,

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } -1 < x < 1.$$

Now integrate the equation $f'(x) = \dfrac{1}{1-x}$,

$$f(x) = \int \frac{1}{1-x}\,dx = -\ln(1-x) + C \quad \text{for } -1 \le x < 1.$$

To determine C, evaluate the equation at $x = 0$,

$$0 = f(0) = -\ln(1-0) + C \implies C = 0$$

Conclusion: $\displaystyle f(x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \text{ for } -1 \le x < 1.$